PROOF OF THE PENROSE CONJECTURE FOR MAXIMALLY PARAMETERIZED INITIAL DATA SETS

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Abstract. We prove the spacetime Penrose inequality for asymptotically flat initial data sets (M^n, g, k) in all dimensions $n \geq 3$, under the dominant energy condition, for data admitting the 'maximally parameterized form' defined in equation (4.7). In this case, the second fundamental form k is locally parameterized by $n(n+1)/2$ functions, giving the same formal local functional degrees of freedom as an arbitrary symmetric 2-tensor.

The proof is based on a new 1-parameter family of generalized Schoen–Yau identities and their gauge-invariant specialization, inspired by the generalized Jang framework of Bray–Khuri [8, 9]. For maximally parameterized data, these identities produce a Riemannian comparison metric

$$\bar{g} = g + \beta \otimes \beta$$

with nonnegative scalar curvature $R_{\bar{g}} \geq 0$. Under a natural nondegeneracy condition (5.7) at the apparent horizon, the comparison metric admits a smooth minimal completion up to ∂M^n. Together with suitable asymptotic control, this reduces the Penrose inequality to its Riemannian counterpart. The all-dimensional conclusion uses the Riemannian Penrose theorem of Bi–Zhu [6], together with a minimizing enclosure argument that allows singular boundaries.

More generally, in the spirit of [8, 9], we note that the Penrose conjecture follows whenever a certain system of PDEs (3.18) involving k can be solved, in which case our new identity (3.15) immediately implies $R_{\bar{g}} \geq 0$. A natural problem is thus to regularize our degenerate system to prove a general existence theory, thereby proving the Penrose conjecture. Essentially, we believe this is what the OpenAI paper [27] posted last week on October 6, 2026 claims to have done.

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1. Introduction

The Penrose inequality [30] in mathematical general relativity conjectures that asymptotically flat initial data sets for the Einstein field equations, satisfying the dominant energy condition, must have ADM mass bounded below by a precise function of the total area of the black holes it contains.

The physical heuristic behind the conjecture combines several basic principles of black hole dynamics. The weak cosmic censorship conjecture [29] predicts that singularities produced by gravitational collapse are hidden behind an event horizon. The Hawking area theorem [20], assuming the appropriate energy condition, implies that the area of the event horizon is nondecreasing toward the future. One further expects an isolated black hole spacetime to radiate away its nonstationary degrees of freedom and approach a stationary black hole; in four-dimensional vacuum general relativity, the black hole uniqueness theorems [14] identify the stationary end state with a Kerr black hole. Since energy may be radiated to null infinity, the mass of the final black hole is no greater than the initial ADM mass, while its horizon area is no smaller than the relevant initial horizon area. Combining these facts with the Kerr mass–area relation leads to the Penrose inequality. In its initial data formulation, the conjecture takes the following form.

Conjecture 1.1 (Penrose [30])**.** *Let (M^n, g, k), $n \geq 3$, be a complete asymptotically flat initial data set satisfying the dominant energy condition $\mu \geq |J|_g$, with compact outermost apparent horizon boundary ∂M^n. Then*

$$m_{\mathrm{ADM}} \geq \frac{1}{2}\left(\frac{\mathcal{A}_g}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}, \tag{1.1}$$

where $\omega_{n-1} = |\mathbb{S}^{n-1}|$, m_{ADM} is the ADM mass, and $\mathcal{A}_g$ is the infimum of the areas of smooth compact hypersurfaces enclosing ∂M^n. Moreover, equality in (1.1) *holds if and only if (M^n, g, k) embeds isometrically into the Schwarzschild spacetime with induced metric g and second fundamental form k.*

The Penrose inequality may be viewed as a refinement of the *positive mass theorem* (PMT) [11, 12, 22, 31–33], which establishes the energy-linear momentum inequality $E_{\mathrm{ADM}} \geq |P_{\mathrm{ADM}}|$. In dimension three, equality characterizes spacelike slices of Minkowski space; in higher dimensions the rigidity statement depends on the asymptotic hypotheses, as discussed in [22, Theorem 1.2 and Remark 1.3]. The Penrose inequality strengthens this to a sharp lower bound determined by the black hole geometry. Note that using the area of an outermost apparent horizon instead of the minimal enclosing area is not valid in general, as there are counterexamples [5].

In the time-symmetric case $k = 0$, the dominant energy condition reduces to nonnegative scalar curvature, the apparent horizon reduces to a minimal surface, and the conjecture becomes the *Riemannian Penrose inequality* (RPI). This was proven by Huisken–Ilmanen [23] for $n = 3$ with a connected boundary using inverse mean curvature flow, and by Bray [7] for $n = 3$ without the connectedness assumption using a conformal flow of metrics. Later Bray–Lee [10] extended Bray's result to dimensions $3 \leq n \leq 7$, and more recently Bi–Zhu [6] extended the same approach to all dimensions $n \geq 3$. Recent progress on the spacetime Penrose inequality includes [1, 2, 15, 16, 18, 21, 26].

In the following, we prove a Penrose inequality for a distinguished class of initial data sets, which we call *maximally parameterized initial data*. To each such initial data set, satisfying the dominant energy condition, we associate a Riemannian comparison metric

$$\bar{g} = g + \beta \otimes \beta$$

with nonnegative scalar curvature, where β is a 1-form. The remaining issues associated with reducing the problem to the Riemannian Penrose inequality involve controlling the behavior of $\bar{g}$ at the apparent horizon and at spatial infinity. Under a natural nondegeneracy condition at the horizon, the comparison metric admits a smooth completion with a totally geodesic, hence minimal, inner boundary; under suitable asymptotic control, its ADM energy agrees with that of the original initial data. To ensure $R_{\bar{g}} \geq 0$, we generalize the approach taken by Bray–Khuri [8, 9], which is already a generalization of the Schoen–Yau approach [32] reducing the spacetime PMT to the Riemannian PMT. In particular, we present a 1-parameter family of Schoen–Yau identities, parameterized by $t \in [0, 1]$, that connects the identity established by Bray–Khuri at $t = 0$ to its gauge-invariant specialization at $t = 1$. The latter identity ultimately results in a gauge-invariant Jang zero-divergence system. We will utilize the following naming conventions associated with different values of the parameter:

$t = 0$	Generalized Schoen–Yau identity	(3.4)
$t \in [0, 1]$	New Schoen–Yau identity	(3.11)
$t = 1$	Gauge-invariant Schoen–Yau identity	(3.15)

The guiding observation is that, for fixed g and k, the left-hand side of the generalized Schoen–Yau identity (the scalar curvature of $\bar{g}$) depends on the graph variables only through $\beta = \phi\, df$, since $\bar{g} = g + \beta \otimes \beta$. We therefore seek a reformulation in which *each term* on the right-hand side has the same invariance. This leads to the new Schoen–Yau identity for general t, with the desired termwise invariance at $t = 1$ on $\{\beta \neq 0\}$; at zeros of β we use the specified smooth intrinsic pair (ν, b) as explained in Section 3. Under the dominant energy condition, the $t = 1$ identity gives $R_{\bar{g}} \geq 0$ when $\mathrm{I} = 0$ and $\mathrm{II} = 0$, the boxed system (3.18). Any prescribed k admitting such a solution, with the global completion and asymptotic properties required in Theorem 6.1, consequently satisfies the stated ADM-energy Penrose bound. The maximally parameterized form (4.7), with Q satisfying its defining weighted divergence equation, constructs tensors k for which this system holds. The bound follows for the resulting initial data under those geometric hypotheses.

The following simplified statement records the main consequence of the construction. A more precise version, including explicit boundary-completion hypotheses, is given in Theorem 6.1.

Theorem 1.2 (Main theorem)**.** *Let (M^n, g, k), $n \geq 3$, be an asymptotically flat maximally parameterized initial data set satisfying the dominant energy condition $\mu \geq |J|_g$, with compact marginally outer trapped surface (MOTS) boundary ∂M^n. Assume that the associated comparison metric $\bar{g}$ admits a smooth completion up to ∂M^n with minimal boundary and induced metric $g|_{T\partial M^n}$, is asymptotically flat, and has the same ADM energy as g. Then*

$$E_{\mathrm{ADM}} \geq \frac{1}{2} \left(\frac{\mathcal{A}_g}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}},$$

where $\mathcal{A}_g$ is the infimum of the g-areas of smooth compact hypersurfaces enclosing ∂M^n.

Remark 1.3 (Energy and mass)**.** ADM energy agrees with ADM mass when $P_{\mathrm{ADM}} = 0$. For $E_{\mathrm{ADM}} > |P_{\mathrm{ADM}}|$, an asymptotic rest-frame boost gives the mass bound if the boost and any required approximation preserve the theorem's hypotheses and the minimum-enclosing-area lower bound; cf. [17, Section 6.3] and [1, Section 5].

Two additional issues remain outside the scope of the theorem. First, the completion hypotheses exclude horizon ends for which the comparison metric develops more degenerate, potentially cylindrical, asymptotics; these are discussed in Section 7.1. Second, the phrase 'maximally parameterized' refers to the local functional degree of freedom count of the construction. We do not claim that every global symmetric 2-tensor k admits such a presentation; the corresponding global surjectivity problem is discussed in Section 7.2.

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2. Geometric Preliminaries

2.1. **Initial data, horizons, and the Riemannian Penrose inequality.** Let (M^n, g) be a smooth, connected, oriented, complete Riemannian manifold with boundary ∂M^n, where $n \geq 3$. An *initial data set* (M^n, g, k) consists of (M^n, g) together with a smooth symmetric 2-tensor k. The tensor k represents the second fundamental form associated with an isometric embedding of (M^n, g), as a spacelike slice, into an $n+1$-dimensional spacetime. Throughout this work the dimension superscript n will typically be removed from the manifold notation. We write $M^\circ = M \setminus \partial M$ for the interior.

The *energy density* μ and *momentum density* J are defined by the constraint equations

$$16\pi\mu = R_g + (\mathrm{tr}_g\, k)^2 - |k|_g^2, \tag{2.1}$$

$$8\pi J = \mathrm{div}_g\, k - d(\mathrm{tr}_g\, k). \tag{2.2}$$

The *dominant energy condition* (DEC) is the requirement $\mu \geq |J|_g$ at every point of M. An initial data set (M^n, g, k) will be referred to as *asymptotically flat* if there exists a compact set $K \subset M$ and a diffeomorphism $\varphi : M \setminus K \to \mathbb{R}^n \setminus B_1(0)$ such that in the associated Cartesian coordinates x the metric and second fundamental form satisfy

$$g_{ij} - \delta_{ij} = O_2(r^{-p}), \qquad k_{ij} = O_1(r^{-p-1}), \qquad r = |x|, \tag{2.3}$$

for some $p > (n-2)/2$. Here $O_j(r^{-a})$ includes coordinate derivatives through order j, with the corresponding additional powers of r^{-1}. Furthermore, we will assume that $\mu, |J|_g \in L^1(M)$. For a Riemannian comparison metric, asymptotic flatness includes the same O_2 metric decay and integrable scalar curvature. The *ADM energy* and *ADM linear momentum* of an asymptotically flat initial data set are defined by

$$E_{\mathrm{ADM}} := \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \sum_{i,j=1}^{n} \big(\partial_j g_{ij} - \partial_i g_{jj}\big) \upsilon^i \, dA, \tag{2.4}$$

$$(P_{\mathrm{ADM}})_i := \lim_{r\to\infty} \frac{1}{(n-1)\omega_{n-1}} \int_{S_r} \sum_{j=1}^{n} \big(k_{ij} - (\mathrm{tr}_g\, k) g_{ij}\big) \upsilon^j \, dA, \tag{2.5}$$

where S_r is the coordinate sphere of radius r, and υ is the unit outer normal. The stated decay and integrability conditions give well-defined ADM quantities, see [4, 13]. For initial data satisfying the hypotheses of the positive mass theorem, the ADM energy–momentum vector is future causal, so that

$$E_{\mathrm{ADM}} \geq |P_{\mathrm{ADM}}|.$$

The corresponding ADM mass is then

$$m_{\mathrm{ADM}} := \sqrt{E_{\mathrm{ADM}}^2 - |P_{\mathrm{ADM}}|^2}. \tag{2.6}$$

A smooth, closed, two-sided hypersurface $\Sigma^{n-1} \subset (M, g, k)$ with outward-pointing unit normal ν is called a *marginally outer trapped surface* (MOTS) if $H_\Sigma + \mathrm{tr}_\Sigma\, k = 0$, and a *marginally inner trapped surface* (MITS) if $H_\Sigma - \mathrm{tr}_\Sigma\, k = 0$, where H_Σ is the mean curvature of Σ in (M, g) (positive for a round sphere in flat $\mathbb{R}^n$) and $\mathrm{tr}_\Sigma\, k = \mathrm{tr}_g\, k - k(\nu, \nu)$ is the trace of k restricted to Σ. In the time-symmetric case $k = 0$, both MOTS and MITS reduce to minimal surfaces ($H_\Sigma = 0$), and the Penrose conjecture reduces to the Riemannian Penrose inequality. We will apply the following result in the final step of our proof.

Theorem 2.1 (Riemannian Penrose inequality; Bray [7], Bray–Lee [10], Bi–Zhu [6])**.** *Let (M^n, g), $n \geq 3$, be asymptotically flat with $R_g \geq 0$, complete up to a compact almost-minimizing Caccioppoli boundary Σ whose regular part is minimal. Assume that Σ is its own outermost g-minimizing enclosure and that g admits a $C^{2,\alpha}$ Riemannian extension across Σ, for some $0 < \alpha < 1$, as in* [6, Theorem 1.2]*. Then*

$$E_{\mathrm{ADM}} \geq \frac{1}{2}\left(\frac{\mathcal{H}_g^{n-1}(\Sigma)}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$

where $\mathcal{H}_g^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure induced by g. Equality holds if and only if (M^n, g) is isometric to a Riemannian Schwarzschild exterior.

2.2. **Integrable 1-forms and Foliation Geometry.** Let $U \subset M^\circ$ be an open set on which a smooth unit 1-form ν satisfies

$$|\nu|_g = 1, \qquad \nu \wedge d\nu = 0.$$

By the Frobenius theorem, the distribution $\ker \nu$ is locally tangent to a codimension-one foliation. Thus, after possibly shrinking U, there exists a smooth local defining function r with $dr \neq 0$ such that

$$\nu = \frac{dr}{|dr|_g}.$$

We write

$$\lambda = |dr|_g, \qquad \Sigma_r = \{r = \mathrm{const}\}$$

for the associated local foliation.

Associated to ν and its level-set foliation we define the *leafwise metric* $\gamma_{ij} = g_{ij} - \nu_i\nu_j$, the *geodesic curvature* $\kappa^i = \nu^j \nabla_j \nu^i$ of the ν-integral curves (which satisfies $g(\kappa, \nu) = 0$), the *mean curvature* $H_\Sigma = \mathrm{div}_g(\nu)$ of the leaves, and the *second fundamental form*

$$P_{ij} = \tfrac{1}{2}(\nabla_i \nu_j + \nabla_j \nu_i) - \tfrac{1}{2}(\nu_i \kappa_j + \nu_j \kappa_i). \tag{2.7}$$

The tensor P is symmetric, satisfies $P\nu = 0$, and has trace $\operatorname{tr}_g P = H_\Sigma$. For a smooth function ψ on M, we denote its leafwise differential by

$$d_\Sigma \psi \coloneqq d\psi - \nu(\psi)\nu.$$

Equivalently, $d_\Sigma\psi$ is the restriction of $d\psi$ to the tangent spaces of the leaves, extended by zero on ν. For a vector field X tangential to the leaves (i.e. $g(X,\nu) = 0$), the ambient and leaf divergences are related by

$$\operatorname{div}_\Sigma(X) = \operatorname{div}_g(X) + g(X,\kappa). \tag{2.8}$$

The following identity relates the geodesic curvature to the foliation function r, and becomes useful later on.

Lemma 2.2. *Let $\nu = dr/|dr|_g$ and $\lambda = |dr|_g$. Then*

$$\kappa = d_\Sigma \log\lambda. \tag{2.9}$$

Proof. Write $\nu_i = \partial_i r/\lambda$. Differentiating:

$$\nabla_j \nu_i = \frac{\nabla_j\nabla_i r}{\lambda} - \frac{\nu_i\,\partial_j\lambda}{\lambda}.$$

Contracting with ν^j and using $\nu^j\nabla_j\nabla_i r = \partial_i\lambda$ (which follows from $\nu^j\nabla_j\nabla_i r = \nu^j\nabla_i\nabla_j r = \nabla_i(\nu^j\nabla_j r) - \nabla_j r\,\nabla_i\nu^j = \nabla_i\lambda - \lambda\nu^j\nabla_i\nu_j = \nabla_i\lambda$, since $\nu^j\nabla_i\nu_j = \frac{1}{2}\nabla_i|\nu|_g^2 = 0$):

$$\kappa_i = \partial_i\log\lambda - \nu_i\,\nu(\log\lambda) = (d_\Sigma\log\lambda)_i. \qquad \square$$

Finally, let $b\colon M \to \mathbb{R}$ be a smooth scalar function and set

$$W = (1+b^2)^{1/2}, \qquad \bar g = g + b^2\,\nu\otimes\nu.$$

We refer to $\bar g$ as the associated Riemannian comparison metric. Its inverse metric and volume element are

$$\bar g^{ij} = g^{ij} - v^i v^j, \qquad \sqrt{\det\bar g} = W\sqrt{\det g},$$

where

$$v = \frac{b}{W}\nu.$$

In particular,

$$|v|_g = \frac{|b|}{W} < 1$$

whenever b is finite.

3. The New Schoen–Yau Identity and Its Gauge-Invariant Form

3.1. **The Generalized Schoen–Yau Identity ($t = 0$).** We review the Jang deformation, originally applied by Schoen–Yau to successfully reduce the spacetime PMT to the Riemannian PMT and later adapted by Bray–Khuri in an attempt to do the same for the Penrose inequality. The proof of Theorem 6.1 relies on a modification of the latter strategy, resulting in the new Schoen–Yau identity and its gauge-invariant $t = 1$ specialization, introduced below.

Given asymptotically flat initial data (M, g, k) satisfying the DEC, consider smooth functions $f, \phi : M \to \mathbb{R}$ with $\phi > 0$ and construct the deformed metric

$$\bar g \coloneqq g + \phi^2\,df\otimes df. \tag{3.1}$$

Using this metric, construct the warped product spacetime $(\mathbb{R}\times M, -\phi^2\,dt^2+\bar{g})$. The Riemannian manifold (M,g) embeds as a graph in $(\mathbb{R}\times M, -\phi^2\,dt^2+\bar{g})$ with metric $g=\bar{g}-\phi^2\,df\otimes df$ and second fundamental form

$$h=\frac{\phi\nabla_g^2 f+d\phi\otimes df+df\otimes d\phi}{\sqrt{1+\phi^2|df|_g^2}}. \tag{3.2}$$

Define the difference tensor

$$\delta(0)\coloneqq h-k. \tag{3.3}$$

The choice of notation here will become clear in the next section when we deform this difference to derive the new Schoen–Yau identity for general t. The scalar curvatures of the metrics $\bar{g}$ and g are related by the generalized Schoen–Yau identity, developed by Bray–Khuri [8,9].

Theorem 3.1 (Generalized Schoen–Yau identity, $t=0$)**.** *For any smooth f, $\phi>0$ on (M,g,k), define the difference $\delta(0)\coloneqq h-k$ and the vector $v\coloneqq\frac{\phi\,\nabla f}{\sqrt{1+\phi^2|df|_g^2}}$. Then the following identity holds for $\bar{g}=g+\phi^2 df\otimes df$:*

$$\boxed{\begin{aligned}&\textbf{\textit{Generalized Schoen–Yau identity}}\quad (t=0)\\ R_{\bar{g}}&=16\pi\big(\mu-J(v)\big)+|\delta(0)|_{\bar{g}}^2+2|\delta(0)(v,\cdot)|_{\bar{g}}^2\\ &\quad-\frac{2}{\phi}\operatorname{div}_{\bar{g}}\big(\phi\,\delta(0)(v,\cdot)\big)+\big(\operatorname{tr}_{\bar{g}}\,\delta(0)\big)^2\\ &\quad+2\big(\operatorname{tr}_{\bar{g}}\,k+k(v,v)\big)\operatorname{tr}_{\bar{g}}\,\delta(0)+2v\big(\operatorname{tr}_{\bar{g}}\,\delta(0)\big).\end{aligned}} \tag{3.4}$$

The proof is a long calculation that Bray and Khuri carry out in the appendices of their paper. Suppose we choose f and ϕ to satisfy the *generalized Jang equation*

$$\operatorname{tr}_{\bar{g}}h=\operatorname{tr}_{\bar{g}}k. \tag{3.5}$$

For such f and ϕ, the generalized Schoen–Yau identity becomes

$$R_{\bar{g}}=16\pi(\mu-J(v))+|\delta(0)|_{\bar{g}}^2+2|\delta(0)(v,\cdot)|_{\bar{g}}^2-\frac{2}{\phi}\operatorname{div}_{\bar{g}}(\phi\,\delta(0)(v,\cdot)). \tag{3.6}$$

The trick to make this applicable to geometric inequalities is the proper choice of second equation for f and ϕ. The choice applicable to the PMT is simply $\phi=1$. In this case, the identity simplifies to

$$R_{\bar{g}}\geq-2\operatorname{div}_{\bar{g}}\big(\delta(0)(v,\cdot)\big). \tag{3.7}$$

This is *almost* nonnegative scalar curvature. Schoen–Yau then complete the argument by conformally changing the metric $\bar{g}$ to a metric of zero scalar curvature (to which the Riemannian PMT applies) and ADM energy no greater than that of $\bar{g}$.

Several complications arise when one tries to apply this approach to the spacetime Penrose inequality. First of all, it is unclear what the right choice for ϕ is in this case. One cannot use $\phi=1$ since, in the case of equality, we want (M,g,k) to embed isometrically into the

Schwarzschild spacetime, which should require $\phi = \sqrt{1 - \frac{2m}{r^{n-2}}}$. The most obvious choice would be that ϕ is chosen to satisfy

$$\frac{1}{\phi} \operatorname{div}_{\bar{g}}\big(\phi\, \delta(0)(v, \cdot)\big) = 0. \tag{3.8}$$

Equation (3.8), together with the generalized Jang equation (3.5), forms the Jang zero-divergence system introduced by Bray–Khuri [8, 9]. However, Jaracz [25] constructed spherically symmetric initial data sets for which no smooth radial solution of this system exists with positive exterior lapse and the asymptotics required for the Penrose application. His theorem does not rule out nonsymmetric solutions.

The key observation that allows us to apply this program to prove Theorem 6.1 below, is that the left-hand side of (3.4) depends on the 1-form $\beta = \phi\, df$ rather than the functions f and ϕ independently. In particular, changing $\tilde{f} = F(f)$ and $\tilde{\phi} = \frac{\phi}{F'(f)}$ with $F' > 0$ leaves β invariant and thus (3.4) still holds. This motivates the introduction of a 1-parameter family of deformed difference tensors. The special value $t = 1$ will eliminate the dependence on the individual graph variables f and ϕ, on the regular graph set, and leads to a gauge-invariant formulation of the Jang zero-divergence system.

3.2. **The New Schoen–Yau Identity (General t).** The observation in the previous section allows us to extend the generalized Schoen–Yau identity of Bray–Khuri to the new Schoen–Yau identity, a 1-parameter family.

Suppose throughout that $f, \phi : M \to \mathbb{R}$ are smooth functions on an initial data set (M, g, k), with $\phi > 0$. We can define the 1-form

$$\beta = \phi\, df$$

as above and note that $\beta \wedge d\beta = 0$; therefore β is integrable. We can normalize

$$\nu = \frac{\beta}{|\beta|_g}$$

wherever $\beta \neq 0$. Set $b = |\beta|_g = \phi |df|_g$ and $W = \sqrt{1 + b^2} = \sqrt{1 + \phi^2 |df|_g^2}$. We define

$$\sigma \coloneqq k(\nu, \nu) - h(\nu, \nu) \tag{3.9}$$

and the deformed difference tensor

$$\delta(t) \coloneqq h - k + t\sigma\, \nu \otimes \nu. \tag{3.10}$$

Thus $\delta(t)$ deforms the normal–normal component of the difference $h - k$. The new Schoen–Yau identity below demonstrates that we can relate the scalar curvature of the generalized Jang metric

$$\bar{g} = g + \beta \otimes \beta$$

to the deformed difference tensor $\delta(t)$ and the matter term $\mu - J(v)$. The interval $t \in [0, 1]$ gives the interpolation described in the introduction; the identity itself holds for every constant $t \in \mathbb{R}$.

Theorem 3.2 (New Schoen–Yau identity, general t)**.** *For every constant $t \in \mathbb{R}$ and any smooth functions f and $\phi > 0$ on (M, g, k), the following identity holds on the regular graph set $U = \{df \neq 0\} = \{\beta \neq 0\}$, where ν, σ, and $\delta(t)$ are defined:*

$$\boxed{\begin{aligned} &\textbf{\textit{New Schoen–Yau identity}} \quad (\textit{general } t) \\ R_{\bar g} &= 16\pi\big(\mu - J(v)\big) + |\delta(t)|^2_{\bar g} + 2|\delta(t)(v,\cdot)|^2_{\bar g} \\ &\quad - \frac{2}{\phi}\operatorname{div}_{\bar g}\big(\phi\,\delta(t)(v,\cdot)\big) + \frac{2\sigma^2 t(1-t)}{W^2} \\ &\quad + \big(\operatorname{tr}_{\bar g}\delta(t)\big)^2 + 2\big(\operatorname{tr}_{\bar g} k + k(v,v)\big)\operatorname{tr}_{\bar g}\delta(t) \\ &\quad + 2v\big(\operatorname{tr}_{\bar g}\delta(t)\big). \end{aligned}} \tag{3.11}$$

At $t = 0$, the correction term vanishes and $\delta(0) = h - k$, recovering the generalized Schoen–Yau identity of Theorem 3.1.

Proof. The proof is obtained by differentiating the right-hand side of (3.11) with respect to t and showing that it is independent of t. The complete calculation is given in Appendix A. □

3.3. **The Gauge-Invariant Schoen–Yau Identity ($t = 1$).** On $\{\beta \neq 0\}$, the gauge-invariant Schoen–Yau identity ($t = 1$) and its separate terms depend only on g, k, and $\beta = \phi\, df$, so they are invariant under graph reparametrization. To state the intrinsic identity also at zeros, specify a smooth integrable unit 1-form ν and a smooth signed function b, and put $W = \sqrt{1 + b^2}$. Define directly

$$\Delta := \frac{b}{W}P - k + k(\nu,\nu)\nu\otimes\nu + \frac{1}{W}\big(\nu\otimes d_\Sigma b + d_\Sigma b\otimes\nu\big), \tag{3.12}$$

where P is the leafwise second fundamental form extended by zero in the normal direction. This tensor is smooth for the specified (ν, b), including at $b = 0$, and agrees with $\delta(1)$ on the regular graph set. At zeros of β, the separate terms can depend on the chosen foliation; they are not determined by β alone.

Theorem 3.3 (Gauge-invariant Schoen–Yau identity, $t = 1$)**.** *Let (M, g, k) be an initial data set. Let ν be a smooth 1-form satisfying*

$$|\nu|_g = 1, \qquad \nu \wedge d\nu = 0,$$

and let Σ denote the leaves of the codimension-one foliation tangent to $\ker\nu$. Denote the mean curvature of the leaves by

$$H_\Sigma = \operatorname{div}_g(\nu) \tag{3.13}$$

and the geodesic curvature by

$$\kappa = \nabla_\nu \nu. \tag{3.14}$$

Suppose b is a smooth scalar function and define $W = (1 + b^2)^{1/2}$ and $\bar g = g + b^2\,\nu\otimes\nu$. With Δ defined by (3.12), the following identity holds throughout the domain of the smooth pair (ν, b),

including where $b = 0$:

Gauge-invariant Schoen–Yau identity $(t = 1)$

$$R_{\bar{g}} = 16\pi\left(\mu - J\left(\frac{b}{W}\nu\right)\right) + |\Delta|_g^2 - \frac{2}{W}\mathrm{II} + \mathrm{I}^2 + 2\,\mathrm{tr}_g\, k\,\mathrm{I} + \frac{2b}{W}\nu(\mathrm{I}). \tag{3.15}$$

where

$$\mathrm{I} := \frac{b}{W}\,H_\Sigma - \mathrm{tr}_\Sigma\, k \tag{3.16}$$

and

$$\mathrm{II} := 2\left\langle d_\Sigma b, \Delta(\nu,\cdot)|_\Sigma\right\rangle_g + b\left(\mathrm{div}_\Sigma\big(\Delta(\nu,\cdot)|_\Sigma\big) - 2\Delta(\nu,\kappa)\right). \tag{3.17}$$

Every term in (3.15) *depends only on g, k, ν, and b.*

Proof. Where $b \neq 0$, specialize the new Schoen–Yau identity to $t = 1$ and rewrite it in terms of the intrinsic pair (ν, b), as in Appendix A.3. At a point p with $b(p) = 0$, apply this already established identity to $b_\varepsilon = b + \varepsilon$ on a neighborhood of p on which $b_\varepsilon \neq 0$, and let $\varepsilon \to 0$. All intrinsic terms, and the scalar curvature of $g + b_\varepsilon^2 \nu \otimes \nu$, converge smoothly at p. This proves the identity even when the zero set of b has interior. □

Remark 3.4. Although the expression for II is somewhat lengthy, it only depends on the mixed normal–tangential component of the discrepancy tensor Δ. In fact,

$$\Delta(\nu,\cdot)|_\Sigma = \frac{d_\Sigma b}{W} - k(\nu,\cdot)|_\Sigma.$$

This observation will be useful in the next section: the maximally parameterized form of k is chosen precisely so that this mixed component is represented by the tangential 1-form Q, reducing the condition $\mathrm{II} = 0$ to a single weighted leafwise divergence equation.

The gauge-invariant Schoen–Yau identity motivates the following *gauge-invariant Jang zero-divergence system.* We use the notation I and II throughout for the two expressions defined above. The system $\mathrm{I} = 0$, $\mathrm{II} = 0$ is explicitly

Gauge-invariant Jang zero-divergence system

$$\begin{aligned} \mathrm{I} &= \frac{b}{W}H_\Sigma - \mathrm{tr}_\Sigma\, k = 0, \\ \mathrm{II} &= 2\left\langle d_\Sigma b, \Delta(\nu,\cdot)|_\Sigma\right\rangle_g \\ &\quad + b\left(\mathrm{div}_\Sigma\big(\Delta(\nu,\cdot)|_\Sigma\big) - 2\Delta(\nu,\kappa)\right) = 0. \end{aligned} \tag{3.18}$$

We refer to $\mathrm{I} = 0$ as the *trace equation* and to $\mathrm{II} = 0$ as the *zero-divergence equation.* When both equations hold, the gauge-invariant Schoen–Yau identity reduces to

$$R_{\bar{g}} = 16\pi\left(\mu - J\left(\frac{b}{W}\nu\right)\right) + |\Delta|_g^2.$$

Since

$$\left|\frac{b}{W}\nu\right|_g < 1$$

where b is finite, the dominant energy condition implies

$$R_{\bar{g}} \geq 0.$$

Thus, solving the gauge-invariant Jang zero-divergence equations produces a Riemannian comparison metric

$$\bar{g} = g + b^2 \nu \otimes \nu$$

with nonnegative scalar curvature.

In the following section, we provide an explicit parameterization of k satisfying $\mathrm{I} = 0$ and $\mathrm{II} = 0$, motivating the distinguished class of initial data for which the Penrose inequality is satisfied.

4. Maximally Parameterized Initial Data

The equations $\mathrm{I} = 0$ and $\mathrm{II} = 0$ constrain only two pieces of k relative to the foliation: its tangential trace and one scalar condition on its mixed normal–tangential component. This makes the system particularly amenable to an explicit parametrization.

Any symmetric 2-tensor decomposes uniquely as

$$k = A\,\nu \otimes \nu + B\,\gamma + C \otimes \nu + \nu \otimes C + \hat{k}, \tag{4.1}$$

where A, B are scalar functions, C is tangential, and $\hat{k}$ is symmetric, tangential, and trace-free with respect to γ. The trace equation $\mathrm{I} = 0$ determines the tangential trace

$$B = \frac{bH_\Sigma}{(n-1)W}. \tag{4.2}$$

For the mixed component, recall that

$$\Delta(\nu, \cdot)|_\Sigma = \frac{d_\Sigma b}{W} - k(\nu, \cdot)|_\Sigma = \frac{d_\Sigma b}{W} - C.$$

We therefore introduce the tangential 1-form

$$Q \coloneqq \Delta(\nu, \cdot)|_\Sigma = \frac{d_\Sigma b}{W} - C. \tag{4.3}$$

Equivalently,

$$C = \frac{d_\Sigma b}{W} - Q. \tag{4.4}$$

With this notation, the zero-divergence equation $\mathrm{II} = 0$ becomes

$$2\langle d_\Sigma b, Q\rangle_g + b\big(\mathrm{div}_\Sigma Q - 2Q(\kappa)\big) = 0. \tag{4.5}$$

These observations motivate the following definition.

Definition 4.1 (Maximally Parameterized Presentation)**.** Let (M, g) be a complete Riemannian manifold. For an open subset $U \subset M^\circ$, a **maximally parameterized presentation** on U consists of data $(\nu, b, A, Q, \hat{k})$ with the following properties.

(1) $\nu \in \Omega^1(U)$ is a smooth 1-form satisfying $|\nu|_g = 1$ and $\nu \wedge d\nu = 0$.

(2) $b, A \in C^\infty(U)$ are arbitrary smooth, signed scalar functions.
(3) $Q \in \Omega^1(U)$ is a smooth 1-form tangent to the leaves of the foliation determined by ν: $Q(\nu) = 0$. Moreover, Q satisfies the weighted leafwise divergence equation

$$2\langle d_\Sigma b, Q\rangle_g + b(\mathrm{div}_\Sigma Q - 2Q(\kappa)) = 0. \tag{4.6}$$

(4) $\hat{k} \in \Gamma(S^2T^*U)$ is a smooth symmetric 2-tensor that is tangential and tangentially trace free: $\hat{k}(\nu, \cdot) = 0$ and $\mathrm{tr}_\Sigma \hat{k} = 0$.

From this data, one can build the symmetric 2-tensor

$$k_{MP} = A\,\nu \otimes \nu + \frac{bH_\Sigma}{(n-1)W}\gamma + \left(\frac{d_\Sigma b}{W} - Q\right) \otimes \nu + \nu \otimes \left(\frac{d_\Sigma b}{W} - Q\right) + \hat{k}, \tag{4.7}$$

where $\gamma := g - \nu \otimes \nu$ is the leafwise metric.

The following proposition demonstrates the utility of maximally parameterized presentations, namely, they yield initial data sets (M, g, k) satisfying the gauge-invariant Jang zero-divergence equations.

Proposition 4.2. *Every maximally parameterized presentation satisfies the gauge-invariant Jang zero-divergence equations*

$$\mathrm{I} = 0, \qquad \mathrm{II} = 0.$$

Proof. Taking the tangential trace of (4.7) gives

$$\mathrm{tr}_\Sigma\, k_{MP} = \frac{bH_\Sigma}{(n-1)W}\,\mathrm{tr}_\Sigma\,\gamma = \frac{bH_\Sigma}{W},$$

since $\mathrm{tr}_\Sigma\,\gamma = n - 1$. Hence

$$\mathrm{I} = \frac{b}{W}H_\Sigma - \mathrm{tr}_\Sigma\, k_{MP} = 0.$$

For the mixed normal–tangential component, (4.7) gives

$$k_{MP}(\nu, \cdot)|_\Sigma = \frac{d_\Sigma b}{W} - Q.$$

Therefore

$$\Delta(\nu, \cdot)|_\Sigma = \frac{d_\Sigma b}{W} - k_{MP}(\nu, \cdot)|_\Sigma = Q.$$

Substituting this into the definition of II yields

$$\mathrm{II} = 2\langle d_\Sigma b, Q\rangle_g + b\big(\mathrm{div}_\Sigma\, Q - 2Q(\kappa)\big),$$

which vanishes by the defining equation for Q. □

We have the following immediate corollary which is essential to the proof of the Penrose inequality given in the forthcoming sections.

Corollary 4.3. *Let $(\nu, b, A, Q, \hat{k})$ be a maximally parameterized presentation on an open set $U \subset M^\circ$, and let $k = k_{MP}$ on U. If the initial data (g, k) satisfy the dominant energy condition on U, then the comparison metric*

$$\bar{g} = g + b^2 \nu \otimes \nu$$

satisfies

$$R_{\bar{g}} \geq 0 \qquad \text{on } U.$$

With these results in hand, we can define the class of initial data for which we can prove the Penrose inequality.

Definition 4.4 (Maximally Parameterized Initial Data)**.** A smooth initial data set (M, g, k) is *maximally parameterized* if there exists a closed set

$$\mathcal{S} \subset M^\circ$$

with empty interior, and a maximally parameterized presentation $(\nu, b, A, Q, \hat{k})$ on $U = M^\circ \setminus \mathcal{S}$ such that:

(1) the 1-form

$$\beta := b\nu$$

extends smoothly across $\mathcal{S}$ to a smooth 1-form on M°;

(2) the tensor determined by the presentation satisfies

$$k_{MP} = k|_U.$$

Remark 4.5. The set $\mathcal{S}$ records singularities which may be present in the foliation from the maximally parameterized presentation, not singularities in the initial data. For multiple boundary components, a foliation by compact leaves connecting separate near-boundary leaves to connected leaves near infinity must undergo topology change; singularities of such a foliation are therefore natural. The scalar curvature argument only needs the presentation on a dense regular set, provided the actual geometric fields extend smoothly.

Remark 4.6. On any regularly foliated neighborhood $U \subset M^\circ \setminus \mathcal{S}$ on which $b \neq 0$, integrability implies the existence of a local defining function r such that

$$\nu = \frac{dr}{|dr|_g}.$$

Writing $\lambda = |dr|_g$, recall from Lemma 2.2 that

$$\kappa = d_\Sigma \log \lambda.$$

Taking the divergence along the leaves of the foliation gives

$$\begin{aligned}
\operatorname{div}_\Sigma \left(\frac{b^2}{\lambda^2} Q \right) &= \frac{1}{\lambda^2} \operatorname{div}_\Sigma (b^2 Q) - \frac{2b^2}{\lambda^2} Q(\kappa) \\
&= \frac{1}{\lambda^2} \left[2b \langle d_\Sigma b, Q \rangle_g + b^2 \operatorname{div}_\Sigma Q - 2b^2 Q(\kappa) \right] \\
&= \frac{b}{\lambda^2} \left[2 \langle d_\Sigma b, Q \rangle_g + b \big(\operatorname{div}_\Sigma Q - 2Q(\kappa) \big) \right] \\
&= 0.
\end{aligned}$$

Thus

$$\operatorname{div}_\Sigma \left(\frac{b^2}{\lambda^2} Q \right) = 0.$$

Therefore, on a contractible open subset of any leaf, the 1-form

$$\frac{b^2}{\lambda^2} Q$$

is co-closed. Equivalently, its Hodge dual is closed. By the Poincaré lemma, there exists an $(n-3)$-form Ψ such that, up to the sign determined by the Hodge-star convention,

$$\frac{b^2}{\lambda^2} Q = *_\Sigma d_\Sigma \Psi.$$

Hence, wherever $b \neq 0$,

$$Q = \frac{\lambda^2}{b^2} *_\Sigma d_\Sigma \Psi.$$

The potential Ψ is determined only up to the addition of a closed $(n-3)$-form. Locally, for $n \geq 4$, this includes the gauge freedom

$$\Psi \longmapsto \Psi + d_\Sigma \chi.$$

When $n = 3$, Ψ is a function and is determined up to an additive constant. The covector Q is intrinsic, while Ψ gives a local parameterization of solutions to the weighted divergence equation.

Remark 4.7. We can give a formal local functional degree-of-freedom count as follows. On a regular chart, the foliation function, b, and A each contribute one degree of freedom. The symmetric tangential trace-free tensor $\hat{k}$ on an $(n-1)$-dimensional leaf contributes

$$\frac{n(n-1)}{2} - 1$$

degrees of freedom. Finally, the tangential 1-form Q has $n-1$ components and satisfies one scalar first-order equation, so formally it contributes $n-2$ degrees of freedom. Thus the total is

$$1 + 1 + 1 + \left(\frac{n(n-1)}{2} - 1\right) + (n-2) = \frac{n(n+1)}{2},$$

which is the number of independent components of a symmetric 2-tensor k on an n-dimensional manifold. It is important to note that this is only a formal local count, not a global surjectivity statement. This count is what motivates the name 'maximally parameterized'.

Theorem 4.8. *Let (M, g, k) be a maximally parameterized initial data set satisfying the dominant energy condition. Then*

$$\bar{g} = g + \beta \otimes \beta$$

defines a smooth Riemannian metric on M° satisfying

$$R_{\bar{g}} \geq 0.$$

Proof. Let $\mathcal{S} \subset M^\circ$ be the closed singular set appearing in the definition of a maximally parameterized initial data set, and set

$$U = M^\circ \setminus \mathcal{S}.$$

On U, the maximally parameterized presentation satisfies the gauge-invariant Jang zero-divergence equations. Hence, by Corollary 4.3,

$$R_{\bar{g}} \geq 0 \qquad \text{on } U.$$

By definition, the 1-form

$$\beta = b\nu$$

extends smoothly across $\mathcal{S}$. Therefore

$$\bar{g} = g + \beta \otimes \beta$$

extends smoothly across $\mathcal{S}$ as a Riemannian metric on M°. Since $\mathcal{S}$ has empty interior, U is dense in M°. The scalar curvature of a smooth metric is continuous, and therefore

$$R_{\bar{g}} \geq 0$$

on all of M°. □

5. ADM Energy and Horizon Completion

Theorem 4.8 supplies the primary geometric input for reducing the Penrose inequality for maximally parameterized initial data to the Riemannian Penrose inequality. It remains to understand the behavior of the comparison metric $\bar{g}$ at spatial infinity and at the marginally outer trapped boundary.

Remark 5.1. A sufficient asymptotic hypothesis is

$$g - \delta = O_2(r^{-p}), \qquad \beta = O_2(r^{-q}), \qquad p, q > \frac{n-2}{2}, \qquad R_g \in L^1.$$

Then $\bar{g} - g = O_2(r^{-2q})$, and the difference of the ADM fluxes is $O(r^{n-2-2q}) \to 0$. Moreover, $R_{\bar{g}} - R_g = O(r^{-2q-2})$, which is integrable at infinity; the smooth completion ensures integrability on the remaining compact region. Thus $\bar{g}$ has the required asymptotic regularity and integrable scalar curvature. In particular, if β has compact support, or satisfies this decay, then

$$E_{\mathrm{ADM}}(\bar{g}) = E_{\mathrm{ADM}}(g). \tag{5.1}$$

In order to complete the proof of the Penrose inequality, we must ensure that the deformed metric $\bar{g}$ extends to the boundary of the initial data set and that the resulting boundary is minimal with respect to $\bar{g}$.

Let Σ_r be a regular leaf on which $H_{\Sigma_r} \neq 0$. The trace equation $\mathrm{I} = 0$ gives

$$\frac{\mathrm{tr}_{\Sigma_r} k}{H_{\Sigma_r}} = \frac{b}{W}. \tag{5.2}$$

Recall that the null expansions are

$$\theta_+ = H_{\Sigma_r} + \mathrm{tr}_{\Sigma_r} k, \qquad \theta_- = H_{\Sigma_r} - \mathrm{tr}_{\Sigma_r} k.$$

Using (5.2), define

$$u := W^{-2} = 1 - \left(\frac{\mathrm{tr}_{\Sigma_r} k}{H_{\Sigma_r}}\right)^2 = \frac{\theta_+ \theta_-}{H_{\Sigma_r}^2}. \tag{5.3}$$

Suppose Σ_r approaches a future MOTS Σ_0 with $H_{\Sigma_0} \neq 0$. Then

$$\mathrm{tr}_{\Sigma_0} k = -H_{\Sigma_0},$$

and hence (5.2) gives

$$\frac{b}{\sqrt{1+b^2}} \longrightarrow -1.$$

Since

$$b \longmapsto \frac{b}{\sqrt{1+b^2}}$$

takes values in $(-1,1)$ and approaches -1 only as $b \to -\infty$, we obtain

$$b \to -\infty, \qquad u = W^{-2} \to 0.$$

Similarly, if Σ_r approaches a MITS Σ_0 with $H_{\Sigma_0} \neq 0$, then

$$b \to +\infty, \qquad u \to 0.$$

The induced metric on each leaf Σ_r is unchanged by the deformation, since $\bar{g}|_{T\Sigma_r} = g|_{T\Sigma_r} = \gamma_r$. Moreover, since

$$\bar{g}(\nu,\nu) = 1 + b^2 = W^2,$$

the $\bar{g}$-unit normal to Σ_r is

$$\bar{\nu} = \frac{1}{W}\nu.$$

For tangent vectors $X, Y \in T\Sigma_r$, the corresponding second fundamental forms satisfy

$$\bar{P}(X,Y) = \frac{1}{W}P(X,Y).$$

Indeed, using the Lie-derivative characterization of the second fundamental form,

$$\bar{P}(X,Y) = \frac{1}{2}(\mathcal{L}_{\bar{\nu}}\bar{g})(X,Y) = \frac{1}{2W}(\mathcal{L}_{\nu}g)(X,Y) = \frac{1}{W}P(X,Y).$$

Taking the trace with respect to the common induced metric γ_r therefore gives

$$\bar{H}_{\Sigma_r} = \frac{H_{\Sigma_r}}{W} = H_{\Sigma_r}\sqrt{u}. \tag{5.4}$$

Thus, provided H_{Σ_r} remains bounded as $r \to 0$, we have that $u \to 0$ implies

$$\bar{H}_{\Sigma_r} \longrightarrow 0.$$

This suggests that the limiting horizon should become a minimal hypersurface for the deformed metric. Equation (5.4), however, does not determine whether the limiting leaf occurs at finite or infinite $\bar{g}$-distance.

Choose a smooth local defining function $r \geq 0$ for the boundary on a foliated collar and write

$$g = \lambda(r,x)^{-2}dr^2 + \gamma_r(x), \qquad \nu = \frac{dr}{\lambda(r,x)}. \tag{5.5}$$

Since $1 + b^2 = W^2 = u^{-1}$,

$$\bar{g} = \frac{dr^2}{\lambda(r,x)^2 u(r,x)} + \gamma_r(x). \tag{5.6}$$

The order of vanishing of u is therefore the geometric datum that controls the inner end.

Proposition 5.2 (Simple-zero completion)**.** *Assume that* (5.5) *holds on* $[0,\varepsilon) \times \Sigma_0$*, with* $\lambda > 0$ *and* γ_r *smooth up to* $r = 0$*. Suppose* u *extends smoothly to the closed collar and*

$$u(0,x) = 0, \qquad \partial_r u(0,x) > 0 \tag{5.7}$$

for every $x \in \Sigma_0$*. Then* $(M^\circ, \bar{g})$ *admits a smooth finite-distance completion up to* Σ_0*. The completed boundary is totally geodesic, and its induced metric is* $\gamma_0 = g|_{T\Sigma_0}$*.*

Proof. By the Hadamard lemma, the simple-zero hypothesis gives

$$u(r,x) = r\,a(r,x),$$

where a is smooth and $a(0,x) > 0$. Define the variable $s = \sqrt{r}$ so that (5.6) becomes

$$\bar{g} = \frac{4}{\lambda(s^2,x)^2 a(s^2,x)}\,ds^2 + \gamma_{s^2}(x). \tag{5.8}$$

After shrinking the collar if necessary, we may assume $a > 0$. We regard s as the boundary defining coordinate for the completed manifold; in particular, the smooth structure at the completed inner end is the one determined by (s,x) rather than by (r,x).

In these coordinates the coefficient of ds^2 is smooth and strictly positive at $s = 0$, and hence the boundary lies at finite $\bar{g}$-distance. Indeed, along a curve with x fixed the distance to $s = 0$ is bounded by a constant multiple of

$$\int_0^{s_0} ds < \infty.$$

Every coefficient in (5.8) is smooth up to $s = 0$, so this defines a smooth completion with boundary. In fact, the coefficients are even in s, giving a smooth reflected extension through $s = 0$ as an additional property. Because

$$\partial_s \gamma_{s^2}\big|_{s=0} = 0,$$

the second fundamental form of the hypersurface $\{s = 0\}$ vanishes. Thus the completed boundary is totally geodesic, in particular minimal. Its tangential metric is γ_0, so its area agrees with its original g-area. □

Only this completed exterior is used in the Riemannian Penrose argument; reflection is optional.

Remark 5.3 (Relation with MOTS stability)**.** If the leaves vary with normal speed ψ and L denotes the MOTS stability operator with the convention $\frac{d}{dt}\theta_+|_{t=0} = L\psi$, then at a future MOTS with $H > 0$ equation (5.3) gives

$$\frac{d}{dt}u\bigg|_{t=0} = \frac{2}{H}\,L\psi. \tag{5.9}$$

Thus the simple-zero condition means precisely that the chosen foliation moves strictly into the untrapped region to first order. In particular, a strictly stable MOTS has a positive principal eigenfunction furnishing such a variation; see Andersson–Mars–Simon for the stability-operator framework [3]. This observation does not by itself show that an arbitrary preassigned maximally parameterized foliation is the principal-eigenfunction foliation.

Let $\Sigma_0 = \partial M$ denote the designated compact MOTS boundary. Define the minimum enclosing area by

$$\mathcal{A}_g(\Sigma_0) := \inf\big\{|S|_g \;\big|\; S \subset M^\circ \text{ is a smooth compact hypersurface enclosing } \Sigma_0\big\}. \tag{5.10}$$

Define $\mathcal{A}_{\bar{g}}(\Sigma_0)$ analogously after either smooth minimal completion allowed in Theorem 6.1. Since

$$\bar{g} = g + b^2 \nu \otimes \nu \geq g$$

as quadratic forms in the interior, and the completed boundary has the same induced metric as Σ_0 in g, we have

$$|S|_{\bar{g}} \geq |S|_g \tag{5.11}$$

for every admissible enclosure S. Therefore

$$\mathcal{A}_{\bar{g}}(\Sigma_0) \geq \mathcal{A}_g(\Sigma_0). \tag{5.12}$$

To apply the Riemannian Penrose inequality, we use the standard minimizing-enclosure reduction.

Proposition 5.4 (Enclosing-area reduction in all dimensions)**.** *Let (M^n, g), $n \geq 3$, be asymptotically flat, complete up to its compact smooth minimal boundary Σ_0, and satisfy $R_g \geq 0$. Then*

$$E_{\mathrm{ADM}} \geq \frac{1}{2}\left(\frac{\mathcal{A}_g(\Sigma_0)}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

Proof. Take the outermost perimeter-minimizing enclosure S_* of Σ_0; see [6, Definition 2.2 and Remark 2.3]. One may attach an auxiliary smooth collar behind Σ_0 to formulate the containment constraint and use a large mean-convex coordinate sphere as an outer barrier. Compactness and lower semicontinuity of perimeter give a minimizer; taking the outermost minimizer makes it outer minimizing. The auxiliary extension is only used for this variational formulation; no curvature hypothesis on it enters the argument.

Away from Σ_0, the minimizing frontier has a smooth minimal regular part and a singular set of Hausdorff dimension at most $n-8$ [19, Theorem 2.18]. At contact with the smooth minimal obstacle Σ_0, obstacle regularity and the strong maximum principle give smooth contact; a component may coincide with a component of Σ_0. Thus S_* has precisely the boundary regularity allowed in Theorem 2.1.

Smooth exterior approximation of finite-perimeter enclosures, with any contact portions pushed a small distance outward, gives smooth compact enclosures S_j of Σ_0 such that

$$|S_j|_g \longrightarrow \mathcal{H}^{n-1}_g(S_*).$$

Consequently $\mathcal{A}_g(\Sigma_0) \leq \mathcal{H}^{n-1}_g(S_*)$. This approximation step is needed because the definition of $\mathcal{A}_g$ uses smooth competitors even when S_* is singular. The exterior of S_* containing the asymptotically flat end has unchanged ADM energy and nonnegative scalar curvature. Applying Theorem 2.1 to that exterior yields

$$E_{\mathrm{ADM}} \geq \frac{1}{2}\left(\frac{\mathcal{H}^{n-1}_g(S_*)}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} \geq \frac{1}{2}\left(\frac{\mathcal{A}_g(\Sigma_0)}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}. \qquad \square$$

6. Proof of Penrose Inequality for Maximally Parameterized Data

We now have all of the pieces required to prove the main theorem: a Penrose inequality for maximally parameterized initial data.

Theorem 6.1 (Nondegenerate Penrose inequality for maximally parameterized data)**.** *Let (M^n, g, k), $n \geq 3$, be an asymptotically flat maximally parameterized initial data set satisfying the dominant energy condition, with compact MOTS boundary $\Sigma_0 = \partial M$ (or a MITS boundary after time reversal). Assume the following:*

(1) on a foliated collar of each boundary component,

$$u = (1 + b^2)^{-1}$$

extends smoothly to the boundary and has a simple transverse zero in the sense of Proposition 5.2;

(2) the comparison metric $\bar{g}$ is asymptotically flat and

$$E_{\mathrm{ADM}}(\bar{g}) = E_{\mathrm{ADM}}(g). \tag{6.1}$$

Alternatively, item 1 may be replaced by the assumption that $\bar{g}$ admits a smooth completion up to Σ_0 with minimal boundary and induced metric $g|_{T\Sigma_0}$; item 2 is retained in both cases. Then

$$E_{\mathrm{ADM}}(g) \geq \frac{1}{2}\left(\frac{\mathcal{A}_g(\Sigma_0)}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}. \tag{6.2}$$

Proof. By Theorem 4.8,

$$R_{\bar{g}} \geq 0$$

on M°. Under item 1, Proposition 5.2 gives a smooth totally geodesic completion with induced metric $g|_{T\Sigma_0}$. Under the alternative boundary hypothesis, the smooth minimal boundary and equality of tangential metrics are assumed directly. In both cases the scalar curvature remains nonnegative up to the boundary by continuity.

Applying Proposition 5.4 to the completed metric $\bar{g}$ gives

$$E_{\mathrm{ADM}}(\bar{g}) \geq \frac{1}{2}\left(\frac{\mathcal{A}_{\bar{g}}(\Sigma_0)}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

By the area comparison above,

$$\mathcal{A}_{\bar{g}}(\Sigma_0) \geq \mathcal{A}_g(\Sigma_0),$$

and by the asymptotic hypothesis,

$$E_{\mathrm{ADM}}(\bar{g}) = E_{\mathrm{ADM}}(g).$$

The desired inequality (6.2) now follows. □

7. Rigidity and Further Questions

Our construction of maximally parameterized presentations and initial data isolates several problems that should not be conflated with the scalar curvature calculation.

7.1. **Degenerate horizons and cylindrical ends.** The simple-zero branch of Theorem 6.1 treats the nondegenerate regime; its regular minimal boundary alternative does not address degenerate horizon blow-up. The remaining horizon problem is to classify the metric (5.6) when $du = 0$ at the MOTS. A quadratic zero is the natural cylindrical model, but a proof that an arbitrary degenerate maximally parameterized horizon has the quantitative asymptotics required by a cylindrical-end Penrose theorem is still missing. In dimension three, the theorem of Jaracz [24] provides an appropriate target once those asymptotics and the generalized minimizing condition are verified. In higher dimensions an analogous cylindrical-end statement would need to be supplied or derived from an all-dimensional Riemannian Penrose theorem by a truncation/doubling argument.

Equation (5.9) suggests that this dichotomy may be related to MOTS stability. If the actual foliation lapse ψ satisfies $L\psi > 0$, then the horizon is in the simple-zero regime. The marginal case $L\psi = 0$ is the natural place to expect cylindrical behavior. Turning this observation into a complete boundary classification is a concrete next problem.

7.2. **Global surjectivity of the parametrization.** The count of local functional degrees of freedom does not show that a prescribed global k can always be represented in the form (4.7). The open questions include the existence of a suitable integrable one-form ν, the behavior of its singular set, and the solvability of the weighted equation for Q across leaves of changing topology. Thus 'maximally parameterized' should be read as a local functional statement, not as a theorem that the construction parametrizes all initial data.

7.3. **Nonzero ADM linear momentum.** The Riemannian comparison directly controls E_{ADM}. An ADM mass conclusion for nonzero ADM linear momentum requires an admissible rest-frame boost and any necessary approximation that retain the comparison hypotheses and control minimum enclosing area. Establishing these properties within the maximally parameterized class remains an additional problem.

7.4. **Equality and rigidity.** Suppose equality holds in Theorem 6.1. Then equality must hold at each step of the comparison argument. In particular

$$\mathcal{A}_{\bar{g}}(\Sigma_0) = \mathcal{A}_g(\Sigma_0),$$

and equality holds in the Riemannian Penrose inequality for the exterior comparison geometry. The rigidity clause of [6, Theorem 1.2] identifies this exterior with a Riemannian Schwarzschild exterior in every dimension. In particular, on this exterior region we therefore have

$$R_{\bar{g}} = 0.$$

The gauge-invariant Schoen–Yau identity gives

$$0 = 16\pi\big(\mu - J(v)\big) + |\Delta|_g^2.$$

Since both terms on the right-hand side are nonnegative under the dominant energy condition, it follows that

$$\Delta = 0, \qquad \mu - J(v) = 0, \tag{7.1}$$

on the regular presentation set in the exterior comparison region. At every such point, where b is finite,

$$|v|_g < 1.$$

Together with the dominant energy condition, (7.1) therefore implies

$$\mu = 0, \qquad J = 0.$$

The original constraint densities are smooth, so their vanishing extends across the singular presentation set by continuity. Thus equality forces vacuum data throughout the comparison exterior and vanishing discrepancy wherever its intrinsic presentation is defined.

What remains is a genuine spacetime rigidity problem: one must understand solutions of

$$\Delta = 0$$

over the Schwarzschild comparison geometry, and determine when the original initial data (M, g, k) arise as a spacelike slice of the Schwarzschild spacetime. We do not claim this final reconstruction here.

Appendix A. Derivation of the New Schoen–Yau Identity and Its Gauge-Invariant Form

In this appendix, we give the calculations underlying Theorem 3.2 and the gauge-invariant Schoen–Yau identity (3.15). Let $f, \phi\colon M \to \mathbb{R}$ be smooth with $\phi > 0$, and set

$$\bar{g} = g + \phi^2\, df \otimes df, \qquad W = \sqrt{1 + \phi^2 |df|_g^2}, \qquad v = \frac{\phi \nabla f}{W}.$$

On the set where $df \neq 0$, define

$$\nu = \frac{df}{|df|_g},$$

so that

$$v = \frac{\phi |df|_g}{W}\, \nu.$$

We write

$$k_{\nu\nu} = k(\nu, \nu), \qquad h_{\nu\nu} = h(\nu, \nu),$$

and define

$$\sigma = k_{\nu\nu} - h_{\nu\nu}.$$

Recall that

$$\delta(t) = h - k + t\sigma\, \nu \otimes \nu.$$

It will also be convenient to define

$$H(t) \coloneqq h + t\sigma\, \nu \otimes \nu, \tag{A.1}$$

so that

$$\delta(t) = H(t) - k.$$

Since

$$\bar{g}^{ij} = g^{ij} - v^i v^j,$$

we have

$$\bar{g}^{ij}\nu_j = \frac{\nu^i}{W^2}, \qquad |\nu|^2_{\bar{g}} = \frac{1}{W^2}. \tag{A.2}$$

Moreover,

$$\delta(t)_{\nu\nu} = h_{\nu\nu} - k_{\nu\nu} + t\sigma = (t-1)\sigma. \tag{A.3}$$

A.1. **Variation identities. Identity 1.**

$$W^4 \frac{d}{dt}|\delta(t)|^2_{\bar{g}} = 2\sigma^2(t-1) \tag{A.4}$$

Proof. Since $\bar{g}$ is independent of t and

$$\frac{d}{dt}\delta(t) = \sigma\, \nu \otimes \nu,$$

we compute

$$\begin{aligned} W^4 \frac{d}{dt}|\delta(t)|^2_{\bar{g}} &= 2W^4 \bar{g}^{im}\bar{g}^{jn}\delta(t)_{ij}\frac{d}{dt}\delta(t)_{mn} \\ &= 2\sigma W^4 \bar{g}^{im}\bar{g}^{jn}\delta(t)_{ij}\nu_m\nu_n \\ &= 2\sigma\delta(t)_{\nu\nu} \\ &= 2\sigma^2(t-1), \end{aligned}$$

where we used (A.2) in the third line and (A.3) in the final line. □

Identity 2.

$$W^4 \frac{d}{dt}\left(2|\delta(t)(v,\cdot)|^2_{\bar{g}}\right) = 4\phi^2|df|^2_g\sigma^2(t-1) \tag{A.5}$$

Proof. Since v and $\bar{g}$ are independent of t,

$$\begin{aligned} W^4 \frac{d}{dt}\left(2|\delta(t)(v,\cdot)|^2_{\bar{g}}\right) &= 4W^4\bar{g}^{ij}v^k v^\ell \delta(t)_{ik}\frac{d}{dt}\delta(t)_{j\ell} \\ &= 4\sigma W^4 \bar{g}^{ij}v^k v^\ell \delta(t)_{ik}\nu_j\nu_\ell. \end{aligned}$$

Using

$$v^i = \frac{\phi|df|_g}{W}\nu^i$$

and (A.2), this becomes

$$\begin{aligned} W^4 \frac{d}{dt}\left(2|\delta(t)(v,\cdot)|^2_{\bar{g}}\right) &= 4\phi^2|df|^2_g\sigma\,\delta(t)_{\nu\nu} \\ &= 4\phi^2|df|^2_g\sigma^2(t-1), \end{aligned}$$

as claimed. □

Identity 3.

$$W^4 \frac{d}{dt}\left(\operatorname{tr}_{\bar{g}} H(t)\right)^2 = 2\sigma W^2 \operatorname{tr}_{\bar{g}} h + 2t\sigma^2 \tag{A.6}$$

Proof. By (A.1),

$$\frac{d}{dt}\operatorname{tr}_{\bar g} H(t) = \sigma \bar g^{ij}\nu_i\nu_j = \frac{\sigma}{W^2}.$$

Moreover,

$$\operatorname{tr}_{\bar g} H(t) = \operatorname{tr}_{\bar g} h + \frac{t\sigma}{W^2}.$$

Therefore

$$\begin{aligned} W^4\frac{d}{dt}\left(\operatorname{tr}_{\bar g} H(t)\right)^2 &= 2W^4 \operatorname{tr}_{\bar g} H(t)\frac{\sigma}{W^2} \\ &= 2\sigma W^2\left(\operatorname{tr}_{\bar g} h + \frac{t\sigma}{W^2}\right) \\ &= 2\sigma W^2 \operatorname{tr}_{\bar g} h + 2t\sigma^2. \end{aligned}$$

□

Identity 4.

$$W^4\frac{d}{dt}\left[2k(v,v)\operatorname{tr}_{\bar g}\delta(t)\right] = 2\phi^2|df|_g^2\sigma k_{\nu\nu} \tag{A.7}$$

Proof. Since $k(v,v)$ is independent of t,

$$\begin{aligned} W^4\frac{d}{dt}\left[2k(v,v)\operatorname{tr}_{\bar g}\delta(t)\right] &= 2W^4 k(v,v)\frac{d}{dt}\operatorname{tr}_{\bar g}\delta(t) \\ &= 2W^4 k(v,v)\frac{\sigma}{W^2} \\ &= 2\sigma W^2 k(v,v). \end{aligned}$$

Since

$$v = \frac{\phi|df|_g}{W}\nu,$$

we have

$$k(v,v) = \frac{\phi^2|df|_g^2}{W^2}k_{\nu\nu}.$$

Substitution gives (A.7). □

Identity 5.

$$W^4\frac{d}{dt}\left[\frac{2\sigma^2 t(1-t)}{W^2}\right] = 2\sigma^2 W^2(1-2t) \tag{A.8}$$

Proof. The quantities W and σ are independent of t, and hence

$$W^4\frac{d}{dt}\left[\frac{2\sigma^2 t(1-t)}{W^2}\right] = 2\sigma^2 W^2(1-2t).$$

□

Identity 6.

$$W^2\operatorname{div}_{\bar g}(\phi v) = \operatorname{div}_g(\phi v) - \frac{\phi\, v(\phi^2|df|_g^2)}{2W^2} \tag{A.9}$$

Proof. Using

$$\sqrt{\det \bar{g}} = W\sqrt{\det g}$$

and

$$\bar{g}^{ij} v_j = \frac{v^i}{W^2},$$

we compute

$$\begin{aligned} W^2 \operatorname{div}_{\bar{g}}(\phi v) &= \frac{W^2}{\sqrt{\det \bar{g}}} \partial_i \left(\sqrt{\det \bar{g}}\, \bar{g}^{ij} \phi v_j \right) \\ &= \frac{W}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g}\, \frac{\phi v^i}{W} \right) \\ &= \operatorname{div}_g(\phi v) + W \phi v^i \partial_i \left(\frac{1}{W} \right) \\ &= \operatorname{div}_g(\phi v) - \frac{\phi\, v(W)}{W}. \end{aligned}$$

Since

$$W^2 = 1 + \phi^2 |df|_g^2,$$

we have

$$v(W) = \frac{v(\phi^2 |df|_g^2)}{2W},$$

which gives (A.9). □

Identity 7.

$$\operatorname{div}_g(\phi v) = \phi \operatorname{tr}_g h - \frac{\phi\, v(\phi^2 |df|_g^2)}{2W^2} \tag{A.10}$$

Proof. Since

$$\phi v = \frac{\phi^2 \nabla f}{W},$$

we have

$$\operatorname{div}_g(\phi v) = \frac{\phi^2}{W} \Delta_g f + \frac{2\phi}{W} \langle d\phi, df \rangle_g + \phi^2 \left\langle df, d\left(\frac{1}{W} \right) \right\rangle_g .$$

By the definition of h,

$$\phi \operatorname{tr}_g h = \frac{\phi^2}{W} \Delta_g f + \frac{2\phi}{W} \langle d\phi, df \rangle_g .$$

Therefore

$$\begin{aligned} \operatorname{div}_g(\phi v) &= \phi \operatorname{tr}_g h + \phi^2 \left\langle df, d\left(\frac{1}{W} \right) \right\rangle_g \\ &= \phi \operatorname{tr}_g h - \frac{\phi^2}{2W^3} \left\langle df, d(\phi^2 |df|_g^2) \right\rangle_g . \end{aligned}$$

Finally, using

$$v = \frac{\phi \nabla f}{W},$$

the last term becomes

$$-\frac{\phi\, v(\phi^2|df|_g^2)}{2W^2},$$

which proves the identity. □

Identity 8.

$$W^4\frac{d}{dt}\Big[-\frac{2}{\phi}\operatorname{div}_{\bar g}\big(\phi\delta(t)(v,\cdot)\big)+2v\big(\operatorname{tr}_{\bar g}\delta(t)\big)\Big]=-2\sigma W^2\operatorname{tr}_g h \tag{A.11}$$

Proof. We compute the two terms separately. Since

$$\frac{d}{dt}\delta(t)=\sigma\nu\otimes\nu$$

and

$$\nu(v)=\frac{\phi|df|_g}{W},$$

we have

$$\frac{d}{dt}\big(\delta(t)(v,\cdot)\big)=\sigma v.$$

Therefore

$$\begin{aligned}W^4\frac{d}{dt}&\Big[-\frac{2}{\phi}\operatorname{div}_{\bar g}\big(\phi\delta(t)(v,\cdot)\big)\Big]\\&=-\frac{2W^4}{\phi}\operatorname{div}_{\bar g}(\phi\sigma v)\\&=-\frac{2\sigma W^4}{\phi}\operatorname{div}_{\bar g}(\phi v)-2W^4\langle d\sigma,v\rangle_{\bar g}.\end{aligned}$$

Since

$$\bar g^{ij}v_j=\frac{v^i}{W^2},$$

we have

$$\langle d\sigma,v\rangle_{\bar g}=\frac{v(\sigma)}{W^2}.$$

Hence

$$W^4\frac{d}{dt}\Big[-\frac{2}{\phi}\operatorname{div}_{\bar g}\big(\phi\delta(t)(v,\cdot)\big)\Big]=-\frac{2\sigma W^4}{\phi}\operatorname{div}_{\bar g}(\phi v)-2W^2v(\sigma). \tag{A.12}$$

For the second term,

$$\begin{aligned}W^4\frac{d}{dt}\big[2v\big(\operatorname{tr}_{\bar g}\delta(t)\big)\big]&=2W^4v\left(\frac{\sigma}{W^2}\right)\\&=2W^2v(\sigma)+2\sigma W^4v(W^{-2})\\&=2W^2v(\sigma)-2\sigma\, v(\phi^2|df|_g^2).\end{aligned}$$

Adding this to (A.12) gives

$$W^4 \frac{d}{dt}\Big[-\frac{2}{\phi}\operatorname{div}_{\bar g}\big(\phi\delta(t)(v,\cdot)\big) + 2v\big(\operatorname{tr}_{\bar g}\delta(t)\big)\Big]$$
$$= -\frac{2\sigma W^4}{\phi}\operatorname{div}_{\bar g}(\phi v) - 2\sigma\, v(\phi^2|df|_g^2).$$

Using Identity (A.9),

$$= -\frac{2\sigma W^2}{\phi}\left[\operatorname{div}_g(\phi v) - \frac{\phi\, v(\phi^2|df|_g^2)}{2W^2}\right] - 2\sigma\, v(\phi^2|df|_g^2)$$
$$= -\frac{2\sigma W^2}{\phi}\operatorname{div}_g(\phi v) - \sigma\, v(\phi^2|df|_g^2).$$

Finally, Identity (A.10) gives

$$= -2\sigma W^2 \operatorname{tr}_g h + \sigma\, v(\phi^2|df|_g^2) - \sigma\, v(\phi^2|df|_g^2)$$
$$= -2\sigma W^2 \operatorname{tr}_g h.$$

This proves (A.11). □

A.2. **Proof of the New Schoen–Yau Identity (General t).**

Proof of Theorem 3.2. Let $P(t)$ denote the right-hand side of (3.11). At $t = 0$, we have

$$\delta(0) = h - k,$$

and (3.11) is precisely the generalized Schoen–Yau identity of Bray–Khuri. It therefore suffices to show that

$$P'(t) = 0.$$

Since

$$\delta(t) = H(t) - k,$$

we may rewrite

$$\big(\operatorname{tr}_{\bar g}\delta(t)\big)^2 + 2\big(\operatorname{tr}_{\bar g} k + k(v,v)\big)\operatorname{tr}_{\bar g}\delta(t)$$
$$= \big(\operatorname{tr}_{\bar g} H(t)\big)^2 - \big(\operatorname{tr}_{\bar g} k\big)^2 + 2k(v,v)\operatorname{tr}_{\bar g}\delta(t).$$

The quantities μ, J, v, $\bar g$, k, σ, and W are all independent of t. Using Identities (A.4)–(A.8) and (A.11), we obtain

$$W^4 P'(t) = 2\sigma^2(t-1) + 4\phi^2|df|_g^2\sigma^2(t-1) - 2\sigma W^2 \operatorname{tr}_g h$$
$$+ 2\sigma^2 W^2(1-2t) + 2\sigma W^2 \operatorname{tr}_{\bar g} h + 2t\sigma^2$$
$$+ 2\phi^2|df|_g^2\sigma k_{\nu\nu}.$$

Using

$$W^2 = 1 + \phi^2|df|_g^2$$

and collecting the terms involving σ^2 gives

$$\begin{aligned} W^4 P'(t) = &-2\phi^2|df|_g^2\sigma^2 - 2\sigma W^2 \operatorname{tr}_g h \\ &+ 2\sigma W^2 \operatorname{tr}_{\bar g} h + 2\phi^2|df|_g^2\sigma k_{\nu\nu}. \end{aligned}$$

Since

$$\bar g^{ij} = g^{ij} - v^i v^j,$$

we have

$$\operatorname{tr}_{\bar g} h = \operatorname{tr}_g h - h(v,v).$$

Thus

$$W^4 P'(t) = -2\phi^2|df|_g^2\sigma^2 - 2\sigma W^2 h(v,v) + 2\phi^2|df|_g^2\sigma k_{\nu\nu}.$$

But

$$h(v,v) = \frac{\phi^2|df|_g^2}{W^2} h_{\nu\nu},$$

and hence

$$\begin{aligned} W^4 P'(t) &= -2\phi^2|df|_g^2\sigma^2 + 2\phi^2|df|_g^2\sigma\big(k_{\nu\nu} - h_{\nu\nu}\big) \\ &= -2\phi^2|df|_g^2\sigma^2 + 2\phi^2|df|_g^2\sigma^2 \\ &= 0. \end{aligned}$$

Therefore $P(t)$ is independent of t, and since (3.11) holds at $t = 0$, it holds for every $t \in \mathbb{R}$. □

A.3. **The Gauge-Invariant Schoen–Yau Identity ($t = 1$).** We derive (3.15) locally, first where $b > 0$. Write $\nu = dr/|dr|_g$, set $f = r$ and $\phi = b/|dr|_g > 0$, and obtain

$$\beta = b\nu = \phi\, df, \qquad \nu = \frac{df}{|df|_g}, \qquad b = \phi|df|_g, \qquad W = \sqrt{1+b^2}, \qquad v = \frac{b}{W}\nu.$$

The region where $b < 0$ is handled by simultaneously reversing (ν, b): this leaves β, v, Δ, I, and II unchanged, while P and H_Σ reverse sign and κ is unchanged. Thus the positive-b calculation covers both signs. Zeros are treated at the end using the smooth intrinsic definition (3.12). On this regular graph region put $\Delta = \delta(1)$. The component calculations below identify this with (3.12). By the definition of σ,

$$\Delta(\nu,\nu) = 0. \tag{A.13}$$

Let X, Y be tangent to the leaves of $\ker \nu$. Since

$$df = \frac{b}{\phi}\nu,$$

the second fundamental form h of the graph satisfies

$$h(X,Y) = \frac{b}{W} P(X,Y). \tag{A.14}$$

Indeed, the terms involving $d\phi \otimes df$ vanish on tangent vectors, while

$$\nabla^2 f(X,Y) = |df|_g P(X,Y).$$

For the mixed component,

$$h(\nu, X) = \frac{d_\Sigma b(X)}{W}. \tag{A.15}$$

To see this, write $q = |df|_g$, so that $b = \phi q$. Since $X(f) = 0$,

$$\nabla^2 f(\nu, X) = X(q),$$

and therefore

$$Wh(\nu, X) = \phi X(q) + qX(\phi) = X(\phi q) = X(b).$$

It follows from (A.13) that

$$\operatorname{tr}_{\bar g} \Delta = \operatorname{tr}_g \Delta = \operatorname{tr}_\Sigma \Delta.$$

Using (A.14),

$$\begin{aligned} \operatorname{tr}_{\bar g} \Delta &= \frac{b}{W} H_\Sigma - \operatorname{tr}_\Sigma k \\ &= \mathrm{I}. \end{aligned} \tag{A.16}$$

Similarly, (A.15) gives

$$\Delta(\nu, \cdot)|_\Sigma = \frac{d_\Sigma b}{W} - k(\nu, \cdot)|_\Sigma. \tag{A.17}$$

We next compare the quadratic terms in the new Schoen–Yau identity. Because $\Delta_{\nu\nu} = 0$, decomposing Δ into tangential and mixed components gives

$$|\Delta|^2_{\bar g} + 2|\Delta(v, \cdot)|^2_{\bar g} = |\Delta|^2_g. \tag{A.18}$$

Indeed, if

$$Q = \Delta(\nu, \cdot)|_\Sigma,$$

then

$$|\Delta|^2_{\bar g} = |\Delta|^2_{T\Sigma} + \frac{2}{W^2}|Q|^2_g,$$

while

$$2|\Delta(v, \cdot)|^2_{\bar g} = \frac{2b^2}{W^2}|Q|^2_g.$$

Since $W^2 = 1 + b^2$, their sum is exactly $|\Delta|^2_g$.

We now rewrite the divergence term. Set

$$Q = \Delta(\nu, \cdot)|_\Sigma.$$

Then

$$\Delta(v, \cdot) = \frac{b}{W} Q.$$

On the region where $b \neq 0$, the relation

$$b = \phi |df|_g$$

and Lemma 2.2, applied with $r = f$, give

$$d_\Sigma \log \phi = d_\Sigma \log |b| - \kappa. \tag{A.19}$$

For a tangential vector field X, we have

$$\operatorname{div}_g X = \operatorname{div}_\Sigma X - g(X, \kappa),$$

and, because

$$\sqrt{\det \bar{g}} = W\sqrt{\det g},$$

we also have

$$\operatorname{div}_{\bar{g}} X = \operatorname{div}_g X + X(\log W).$$

Taking

$$X = \frac{b}{W}Q,$$

we obtain

$$\begin{aligned}\frac{1}{\phi}\operatorname{div}_{\bar{g}}\left(\phi\frac{b}{W}Q\right) &= \operatorname{div}_\Sigma\left(\frac{b}{W}Q\right) - \frac{b}{W}Q(\kappa)\\ &\quad + \frac{b}{W}\langle d_\Sigma \log W, Q\rangle_g + \frac{b}{W}\langle d_\Sigma \log\phi, Q\rangle_g.\end{aligned}$$

Expanding the first term and using (A.19), we find

$$\begin{aligned}\frac{1}{\phi}\operatorname{div}_{\bar{g}}(\phi\Delta(v,\cdot)) &= \frac{1}{W}\Big[2\langle d_\Sigma b, Q\rangle_g\\ &\qquad + b\left(\operatorname{div}_\Sigma Q - 2Q(\kappa)\right)\Big]\\ &= \frac{\mathrm{II}}{W}.\end{aligned}$$

The final intrinsic expression II/W is smooth for a specified smooth pair (ν, b). The graph calculation applies where $b \neq 0$; the full intrinsic identity at zeros is justified below.

Finally,

$$\operatorname{tr}_{\bar{g}} k + k(v,v) = \operatorname{tr}_g k, \tag{A.20}$$

since

$$\bar{g}^{ij} = g^{ij} - v^i v^j,$$

and

$$v(\mathrm{I}) = \frac{b}{W}\nu(\mathrm{I}). \tag{A.21}$$

Specializing Theorem 3.2 to $t = 1$ and using (A.16), (A.18), (A.20), and (A.21), together with the divergence calculation above, gives

$$R_{\bar{g}} = 16\pi\left(\mu - J\left(\frac{b}{W}\nu\right)\right) + |\Delta|_g^2 - \frac{2}{W}\mathrm{II} + \mathrm{I}^2 + 2\operatorname{tr}_g k\,\mathrm{I} + \frac{2b}{W}\nu(\mathrm{I}).$$

This is precisely the gauge-invariant Schoen–Yau identity (3.15) on $\{b \neq 0\}$. To include any point p with $b(p) = 0$, replace b locally by $b+\varepsilon$ and evaluate the established identity at p, where $b+\varepsilon \neq 0$. Letting $\varepsilon \to 0$ proves the result at p, because all terms defined from $(\nu, b+\varepsilon)$ converge smoothly. This argument also covers open regions on which $b = 0$, without requiring a graph representation there.

Appendix B. Examples of Maximally Parameterized Initial Data Sets

In this appendix, we give several classes of maximally parameterized initial data sets. The examples range from general graph slices of static spacetimes to nonspherically symmetric black hole slices, and rotating data.

B.1. **Graph slices of static spacetimes.** We begin with a general source of maximally parameterized initial data. Let

$$(\widetilde{M}^{n+1}, \widetilde{g}) = (\mathbb{R} \times N^n, -\phi^2 dt^2 + \bar{g})$$

be a static spacetime, where $\phi > 0$ and $\bar{g}$ are independent of t. Let

$$M_f = \{t = f(x) : x \in N\}$$

be a smooth spacelike graph. Its induced metric is

$$g = \bar{g} - \phi^2 df \otimes df. \tag{B.1}$$

Thus, if

$$\beta = \phi\, df,$$

then

$$\bar{g} = g + \beta \otimes \beta. \tag{B.2}$$

Let k denote the second fundamental form of the graph, with the normal chosen consistently with the graph convention used in Section 3. Then k agrees with the tensor h appearing in the generalized Jang construction.

Proposition B.1 (Static graph slices)**.** *On every open set $U \subset M_f$ on which $df \neq 0$, the induced initial data (U, g, k) admit a maximally parameterized presentation. Moreover, the associated Riemannian comparison metric is precisely the static spatial metric $\bar{g}$.*

Proof. On U, define

$$\nu = \frac{df}{|df|_g}, \qquad b = \phi |df|_g.$$

Then

$$\beta = b\nu = \phi\, df$$

and

$$\bar{g} = g + b^2 \nu \otimes \nu.$$

Since $k = h$, we have

$$\delta(0) = h - k = 0.$$

Furthermore,

$$\sigma = k(\nu, \nu) - h(\nu, \nu) = 0,$$

and hence

$$\Delta = \delta(1) = 0.$$

The identities in Appendix A.3 therefore give

$$\mathrm{tr}_\Sigma\, k = \frac{b}{W} H_\Sigma$$

and

$$k(\nu,\cdot)|_\Sigma = \frac{d_\Sigma b}{W}.$$

Set

$$A = k(\nu,\nu), \qquad Q = 0,$$

and let

$$\hat{k} = k|_{T\Sigma} - \frac{\operatorname{tr}_\Sigma k}{n-1}\gamma$$

be the tangential trace-free part of k. Substituting these quantities into (4.7) recovers k exactly. Thus $(\nu, b, A, 0, \hat{k})$ is a maximally parameterized presentation. □

This construction does not impose any symmetry on f. For example, in Minkowski spacetime

$$\widetilde{g} = -dt^2 + \delta,$$

an arbitrary spacelike graph $t = f(x)$ satisfies

$$g = \delta - df \otimes df, \qquad \bar{g} = \delta.$$

Thus arbitrary nonsymmetric spacelike graphs in Minkowski spacetime give maximally parameterized data whose comparison metric is Euclidean.

If the critical set of f has empty interior, it may be included in the singular set $\mathcal{S}$ of Definition 4.4. In examples in which f is constant on an open set, the presentation may instead be extended across that region whenever the unit foliation form ν admits a smooth continuation there. The examples below will be chosen so that this continuation is explicit.

B.2. **Nonspherically symmetric slices of Schwarzschild spacetime.** We next consider a class which is particularly natural for the Penrose inequality. Let

$$F(r) = 1 - \frac{2m}{r^{n-2}}, \qquad r_+ = (2m)^{1/(n-2)},$$

and write the $(n+1)$-dimensional Schwarzschild metric in the exterior region as

$$\widetilde{g} = -F(r)\,dt^2 + \frac{dr^2}{F(r)} + r^2\gamma_{\mathbb{S}^{n-1}}. \tag{B.3}$$

The associated Riemannian Schwarzschild metric is

$$g_{\mathrm{Sch}} = \frac{dr^2}{F(r)} + r^2\gamma_{\mathbb{S}^{n-1}}. \tag{B.4}$$

To describe slices which cross the future horizon smoothly, introduce the ingoing Eddington–Finkelstein coordinate

$$v = t + r^*, \qquad \frac{dr^*}{dr} = \frac{1}{F}.$$

Then

$$\widetilde{g} = -F\,dv^2 + 2\,dv\,dr + r^2\gamma_{\mathbb{S}^{n-1}}, \tag{B.5}$$

which extends smoothly through $r = r_+$.

Let

$$v = V(r,\omega), \qquad \omega \in \mathbb{S}^{n-1},$$

where V is a smooth function chosen so that the slice is spacelike. No spherical symmetry is assumed. Pulling back (B.5) gives

$$g = -F\,dV \otimes dV + dV \otimes dr + dr \otimes dV + r^2 \gamma_{\mathbb{S}^{n-1}}. \tag{B.6}$$

Writing

$$dV = V_r\,dr + d_\omega V,$$

we obtain

$$\begin{aligned} g = {} & \left(2V_r - FV_r^2\right) dr^2 + (1 - FV_r)\,(dr \otimes d_\omega V + d_\omega V \otimes dr) \\ & + r^2 \gamma_{\mathbb{S}^{n-1}} - F\,d_\omega V \otimes d_\omega V. \end{aligned} \tag{B.7}$$

In particular, for a generic choice of $V(r,\omega)$ the induced metric is not spherically symmetric.

On the exterior $r > r_+$, the same slice may be written as a graph

$$t = f(r,\omega), \qquad f = V - r^*.$$

Since the static lapse is $\phi = \sqrt{F}$, Proposition B.1 applies with

$$\beta = \sqrt{F}\,df.$$

Therefore

$$\bar{g} = g + \beta \otimes \beta = g_{\mathrm{Sch}}. \tag{B.8}$$

Thus every such slice has the same Riemannian Schwarzschild comparison geometry.

We now compute the function

$$u = (1 + b^2)^{-1}$$

which governs the behavior of the comparison metric at the horizon. For a static graph,

$$u = 1 - \phi^2 |df|^2_{\bar{g}}.$$

Using

$$df = \left(V_r - \frac{1}{F}\right) dr + d_\omega V$$

and (B.4), we obtain

$$\begin{aligned} 1 - u &= F|df|^2_{g_{\mathrm{Sch}}} \\ &= (FV_r - 1)^2 + \frac{F}{r^2}|d_\omega V|^2_{\gamma_{\mathbb{S}^{n-1}}}. \end{aligned}$$

Hence

$$u = F\left(2V_r - FV_r^2 - \frac{1}{r^2}|d_\omega V|^2_{\gamma_{\mathbb{S}^{n-1}}}\right). \tag{B.9}$$

It is convenient to write

$$B(r,\omega) = 2V_r - FV_r^2 - \frac{1}{r^2}|d_\omega V|^2_{\gamma_{\mathbb{S}^{n-1}}}, \tag{B.10}$$

so that

$$u = FB.$$

The same quantity has a direct spacetime interpretation at the horizon. Restricting (B.7) to $r = r_+$ gives the matrix

$$\begin{pmatrix} 2V_r & (d_\omega V)_B \\ (d_\omega V)_A & r_+^2(\gamma_{\mathbb{S}^{n-1}})_{AB} \end{pmatrix}.$$

The angular block is positive definite. Its Schur complement is

$$2V_r - \frac{1}{r_+^2}|d_\omega V|^2_{\gamma_{\mathbb{S}^{n-1}}} = B(r_+, \omega).$$

Consequently, the slice is spacelike at the future horizon precisely when

$$B(r_+, \omega) > 0. \tag{B.11}$$

Since

$$F(r_+) = 0, \qquad F'(r_+) = \frac{n-2}{r_+} > 0,$$

equations (B.9) and (B.11) imply

$$u(r_+, \omega) = 0$$

and

$$\partial_r u(r_+, \omega) = F'(r_+)B(r_+, \omega) > 0. \tag{B.12}$$

To verify Proposition 5.2, the defining function must follow the presentation foliation, whose leaves are the level sets of $f = V - r^*$ rather than, in general, the areal spheres. Set

$$c = F'(r_+) > 0, \qquad r^* = c^{-1}\log(r - r_+) + h(r),$$

where h is smooth near r_+, and define

$$\rho = e^{-cf} = (r - r_+)e^{ch(r) - cV(r,\omega)}. \tag{B.13}$$

Then ρ is a smooth boundary defining function with the same leaves as f, and $d\rho$ points toward the exterior at the boundary. Choose

$$\nu = \frac{d\rho}{|d\rho|_g}, \qquad b = -\frac{\sqrt{F}\,|d\rho|_g}{c\rho}.$$

This signed choice satisfies $b\nu = \sqrt{F}\,df$. Furthermore,

$$\frac{u}{\rho} = \frac{F(r)}{r - r_+}\,B(r, \omega)e^{-ch(r) + cV(r,\omega)}$$

extends smoothly and positively to the boundary. Transporting angular coordinates by $\nabla_g\rho/|d\rho|_g^2$ gives an orthogonal collar

$$g = \lambda^{-2}d\rho^2 + \gamma_\rho, \qquad \lambda = |d\rho|_g > 0.$$

Thus $u = 0$ and $\partial_\rho u > 0$ at $\rho = 0$ in the actual foliation coordinates, as required by Proposition 5.2.

The boundary cut

$$\Sigma_0 = \{r = r_+,\ v = V(r_+, \omega)\}$$

is a MOTS. Indeed, the vector field ∂_v is tangent to the future horizon, null there, and orthogonal to Σ_0. The induced metric on the horizon sections is

$$r_+^2 \gamma_{\mathbb{S}^{n-1}},$$

which is independent of v. The null second fundamental form in the ∂_v direction therefore has zero trace. In particular,

$$\theta_+ = 0.$$

Moreover,

$$|\Sigma_0|_g = \omega_{n-1} r_+^{n-1}. \tag{B.14}$$

We next record an explicit infinite-dimensional family satisfying the remaining hypotheses of Theorem 6.1. Choose $a > 0$, $R_2 > r_+$, and a smooth function p on $[r_+, \infty)$ satisfying

$$p(r) = aF(r) \quad \text{near } r_+, \qquad 0 < p(r) < 1 \quad (r_+ < r < R_2), \qquad p(r) = 1 \quad (r \geq R_2).$$

Define a radial reference slice V_0 by

$$V_0'(r) = \frac{p(r)}{F(r)}. \tag{B.15}$$

Near the horizon, $V_0' = a$, so V_0 extends smoothly in Eddington–Finkelstein coordinates. For large r,

$$V_0' = (r^*)',$$

and hence

$$V_0 - r^*$$

is constant. Thus the corresponding spacelike slice agrees with a Schwarzschild $t = \text{const}$ slice near infinity. For the reference slice,

$$B_0 = 2V_0' - FV_0'^2 = \frac{p(2-p)}{F} > 0. \tag{B.16}$$

In particular,

$$B_0(r_+) = 2a.$$

Now let

$$\eta \in C^\infty\big([r_+, \infty) \times \mathbb{S}^{n-1}\big)$$

have support contained in $[r_+, R_1] \times \mathbb{S}^{n-1}$ for some $r_+ < R_1 < R_2$, and define

$$V_\varepsilon = V_0 + \varepsilon\eta. \tag{B.17}$$

Then

$$B_\varepsilon = B_0 + 2\varepsilon(1 - FV_0')\eta_r - F\varepsilon^2\eta_r^2 - \frac{\varepsilon^2}{r^2}|d_\omega \eta|^2. \tag{B.18}$$

Since $B_0 > 0$, it follows that for sufficiently small $|\varepsilon|$,

$$B_\varepsilon > 0.$$

Hence V_ε determines a smooth spacelike slice crossing the future horizon. If

$$\eta(r_+, \omega)$$

is nonconstant, then the slice intersects the horizon in a cut which is not invariant under the standard spherical action in the spacetime, namely

$$v = V_0(r_+) + \varepsilon\eta(r_+, \omega).$$

The induced metric on this cut remains the round metric $r_+^2\gamma_{\mathbb{S}^{n-1}}$. Its angular dependence describes the cut in spacetime and the surrounding initial data, rather than a nonround intrinsic horizon geometry. The family (B.17) thus contains arbitrary sufficiently small angular perturbations with the prescribed support. Near the horizon, the radial reference metric is

$$g_0 = \left(2a - a^2F\right) dr^2 + r^2\gamma_{\mathbb{S}^{n-1}},$$

and therefore

$$H_{\Sigma_0}(g_0) = \frac{n-1}{r_+\sqrt{2a}} > 0.$$

Consequently, for sufficiently small $|\varepsilon|$, continuity gives

$$H_{\Sigma_0}(g) > 0$$

for the nonspherical slices as well.

The presentation extends globally through the asymptotic region. Indeed,

$$f_0' = \frac{p-1}{F} < 0 \qquad (r_+ < r < R_2).$$

On the support of η this derivative is bounded away from zero in absolute value (and tends to $-\infty$ at the horizon), so for sufficiently small $|\varepsilon|$ the perturbed f_r is still negative there. Thus df has no new zeros. On $r < R_2$, use

$$\nu = -\frac{df}{|df|_g}, \qquad b = -\sqrt{F}\,|df|_g;$$

these agree with the preceding horizon presentation. Outside the support of η the foliation is radial, and the same fields are

$$\nu = \frac{dr}{|dr|_g}, \qquad b = \sqrt{F}\,f_0'|dr|_g.$$

The latter expressions extend smoothly across $r = R_2$, with $b = 0$ for $r \geq R_2$. The remaining presentation parameters are obtained from k as in Proposition B.1; they extend by $A = Q = \hat{k} = 0$ on the constant-graph region. The slice is therefore exactly a Schwarzschild $t = \text{const}$ slice near infinity. Hence

$$k = 0$$

near infinity and

$$E_{\mathrm{ADM}}(g) = E_{\mathrm{ADM}}(\bar{g}) = m, \qquad P_{\mathrm{ADM}} = 0.$$

The data are vacuum and therefore satisfy the dominant energy condition. We have thus obtained an infinite-dimensional family of nonspherically symmetric maximally parameterized Schwarzschild slices satisfying the hypotheses of Theorem 6.1.

These slices also have an outer-minimizing horizon. For the radial reference metric $g_0 = B_0\,dr^2 + r^2\gamma_{\mathbb{S}^{n-1}}$, the areal spheres have outward mean curvature

$$H_r(g_0) = \frac{n-1}{r\sqrt{B_0}} > 0.$$

This remains positive for all areal spheres under sufficiently small C^2 perturbations $\varepsilon\eta$: on the compact perturbation region it follows by continuity and a positive lower bound for $H_r(g_0)$, and outside that region the metric is unchanged. Thus, the areal spheres form a mean convex foliation of the exterior with respect to g, which implies that Σ_0 is outer minimizing by the divergence theorem applied to their outward unit normal. We then have

$$\mathcal{A}_g(\Sigma_0) = |\Sigma_0|_g = \omega_{n-1} r_+^{n-1}.$$

Since

$$r_+^{n-2} = 2m,$$

these nonsymmetric slices attain equality:

$$m = \frac{1}{2}\left(\frac{|\Sigma_0|_g}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

This example also explains geometrically the horizon hypothesis used in the main theorem. For smooth spacelike slices crossing the nondegenerate Schwarzschild horizon, the simple zero of u is simply the consequence of the spacelike condition together with the simple zero of $F = \phi^2$, the squared Schwarzschild lapse.

A similar construction can be applied to find maximally parameterized slices of Reissner–Nordström without symmetry assumptions.

B.3. **Rotating data: Boyer–Lindquist slices of Kerr.** The maximally parameterized class also contains natural rotating examples. Consider the Kerr spacetime in Boyer–Lindquist coordinates and write the stationary metric in ADM form

$$\widetilde{g} = -N^2 dt^2 + g_{ij}(dx^i + \zeta^i dt)(dx^j + \zeta^j dt). \tag{B.19}$$

On a $t = \text{const}$ slice, the shift is purely azimuthal:

$$\zeta = \zeta^\varphi(r,\theta)\partial_\varphi.$$

The induced metric g and all of its coefficients are independent of φ.

Let

$$\Sigma_r = \{r = \text{const}\}, \qquad \nu = \frac{dr}{|dr|_g}.$$

The shift ζ is tangent to Σ_r. Since the spacetime is stationary, the second fundamental form of the $t = \text{const}$ slice is, up to the overall sign convention for k,

$$k_{ij} = \frac{1}{2N}\left(D_i\zeta_j + D_j\zeta_i\right).$$

Taking the tangential trace along Σ_r gives

$$\operatorname{tr}_{\Sigma_r} k = \frac{1}{N}\operatorname{div}_{\Sigma_r}\zeta$$

up to the same overall sign. Since ζ has only a φ-component and every coefficient is independent of φ,

$$\operatorname{div}_{\Sigma_r}\zeta = \frac{1}{\sqrt{\det\gamma}}\partial_\varphi\left(\sqrt{\det\gamma}\,\zeta^\varphi\right) = 0.$$

Therefore

$$\operatorname{tr}_{\Sigma_r} k = 0. \tag{B.20}$$

Set

$$b = 0.$$

Then $W = 1$ and the trace equation becomes

$$\mathrm{I} = -\operatorname{tr}_{\Sigma_r} k = 0.$$

Moreover,

$$d_\Sigma b = 0,$$

so the zero-divergence equation $\mathrm{II} = 0$ is automatic.

More explicitly, define

$$A = k(\nu, \nu), \qquad Q = -k(\nu, \cdot)|_{\Sigma_r},$$

and let

$$\hat{k} = k|_{T\Sigma_r}.$$

Equation (B.20) implies that $\hat{k}$ is tangentially trace-free. Substituting into (4.7) gives exactly the Kerr second fundamental form. Thus the standard Boyer–Lindquist Kerr initial data are maximally parameterized.

For nonzero angular momentum, the mixed normal–tangential component of k is nonzero, and hence

$$Q \neq 0$$

in general. The Kerr example therefore gives a natural rotating example in which the Q degrees of freedom in the maximally parameterized presentation are genuinely present.

Here

$$\bar{g} = g$$

because $b = 0$. This example is intended to illustrate the size of the maximally parameterized class rather than the simple-zero horizon mechanism in Theorem 6.1; the presentation above does not produce $u \to 0$ at the Kerr horizon.

Appendix C. AI statement

The gauge-invariant Schoen–Yau identity ($t = 1$) was derived entirely by humans in early February 2026. By mid-February, we began using Claude to help with the algebra (which is quite messy) leading to the new Schoen–Yau identity for general t. We used Claude to verify every identity numerically as well as algebraically, thereby providing a double-check against typos, sign errors, or other mistakes. In addition to being an outstanding assistant, Claude also made some useful mathematical suggestions (as we derived the special form for the second fundamental form k in (4.7)) which warrant a sincere acknowledgement at a minimum. Had a human researcher made similar mathematical suggestions, we would have invited them to be a co-author. A younger researcher would likely have accepted, and a senior human researcher might have declined the offer. The help was appreciated, but not decisive.

We also used Claude to draft ideas in short notes which we shared with each other and friends for discussion purposes. However, this paper was written by humans, using the AI drafted notes

as a reference. Our Claude conversations were conducted under settings that exclude them from model training, and we have felt in control of our user experience throughout.

We also played around with Claude looking for an existence theory to various regularized versions of our equations, which in a best case would result in a full proof of the Penrose conjecture. The resulting discussions got very complicated, resulting in us having less confidence in their correctness. Even if they were correct, Claude would have been the lead author, and we didn't want to put out a paper whose ideas were not optimized both for correctness and presentation. We think this is a great direction for the field and warrants further study by others. We decided the present paper was interesting enough for now.

We are also impressed by the productivity of OpenAI's series of papers [28] on the Penrose conjecture and related topics, but the mode of release concerns us. This massive drop of papers is both good for the field in some ways (new ideas) and bad for the field in other ways (mistakes, confusing exposition, lack of clarity what to do next, massive checking needing to be done, etc.).

For example, we think the present paper clarifies what appears to be the overall direction in the OpenAI paper [27] on the proof of the Penrose conjecture. Our new equations (3.18) are degenerate and so need to be regularized. The OpenAI paper begins with what we recognize as regularized versions of our equations with very little motivation, a bit like a magic trick. This matters for understanding (both human and AI) and because it affects what we should do next. For example, it is still open if our original degenerate system (3.18) of equations has a general weak existence theory. Providing this theory would be nice progress which would make it easier for others to find even more applications.

As completion of this paper neared, we knew that leaving open conjectures that imply the Penrose conjecture would inspire many others to try to finish our program. But we were surprised when an OpenAI paper claiming to finish our program appeared on Tuesday, October 6, 2026 (the paper itself carries an earlier internal date), the week *before* we posted. Under current industry practices, privately shared research can enter AI training data without the authors' knowledge — for example, if any recipient of a draft pastes it into a consumer AI product whose default settings permit training. Authors have no way to know whether this has happened to their work, and no lab can demonstrate that it has not; we regard this as a serious and growing problem for scholarly credit. We do not know whether our privately circulated notes influenced the OpenAI results, and under current practices we cannot know. Labs could resolve such questions by publishing training-data provenance; absent that, authors are left with timelines, so we state ours. To be clear, we claim credit for our program as defined in this paper, which we've been developing since well before February 2026, the date of our first new key result.

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